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1969 J. Phys. A: Gen. Phys. 2 293

(http://iopscience.iop.org/0022-3689/2/3/007)

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MS. received 29*th August* 1968, *in revised form* 9*th January* 1969

Abstract. The Schwinger variational principle, together with arbitrary trial functions, is used to derive exact expressions for the scattering amplitudes arising in multi-channel scattering by separable non-local interactions.

We consider a set of n coupled scattering equations containing non-local potentials only, which we express as a single matrix equation having the form

$$(\nabla^2+\mathbf{k}^2)\mathbf{F}(\boldsymbol{r}) = \int \mathbf{K}(\boldsymbol{r},\boldsymbol{r}')\mathbf{F}(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}' \tag{1}$$

where $\mathbf{k}$ is a diagonal matrix with the wave numbers k_m ($m = 1, ..., n$) for the different channels along the leading diagonal, $\mathbf{K}$ is a $n\times n$ matrix satisfying the condition $\mathbf{K}^*(\boldsymbol{r},\boldsymbol{r}') = \mathbf{K}(\boldsymbol{r}',\boldsymbol{r})$, and $\mathbf{F}$ is a column matrix with elements F_m ($m = 1, ..., n$), where for large r

$$F_m(\boldsymbol{r}) \sim \exp(\mathrm{i}\boldsymbol{k}_1\,.\,\boldsymbol{r})\delta_{1m}+r^{-1}\exp(\mathrm{i}k_m r)f_m(\theta,\phi) \tag{2}$$

f_m being the scattering amplitude for the mth channel.

Then the Schwinger variational expression for the scattering amplitudes takes the form

$$\begin{aligned}\mathbf{f} = -\frac{1}{4\pi}\int\int\tilde{\boldsymbol{\phi}}(\boldsymbol{r})\mathbf{K}(\boldsymbol{r},\boldsymbol{r}')\mathbf{F}(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}'\\ .\left\{\int\int\tilde{\mathbf{F}}(\boldsymbol{r})\mathbf{K}(\boldsymbol{r},\boldsymbol{r}')\mathbf{F}(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}'\right.\\ \left.+\int\int\int\int\tilde{\mathbf{F}}(\boldsymbol{r})\mathbf{K}(\boldsymbol{r},\boldsymbol{r}')\mathbf{G}(\boldsymbol{r}',\boldsymbol{r}'')\mathbf{K}(\boldsymbol{r}'',\boldsymbol{r}''')\mathbf{F}(\boldsymbol{r}''')\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}'\,\mathrm{d}\boldsymbol{r}''\,\mathrm{d}\boldsymbol{r}'''\right\}^{-1}\\ .\int\int\tilde{\mathbf{F}}(\boldsymbol{r})\mathbf{K}(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{\phi}(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}'\end{aligned} \tag{3}$$

which is stationary with respect to small arbitrary variations of the solution $\mathbf{F}$ and its adjoint $\tilde{\mathbf{F}}$. Here $\mathbf{f}$ is a column matrix with elements f_m ($m = 1, ..., n$), $\mathbf{G}$ is a diagonal matrix with the free-particle Green functions for the n channels along the leading diagonal, $\boldsymbol{\phi}$ is a column matrix with elements $\exp(\mathrm{i}\boldsymbol{k}_1.\boldsymbol{r})\delta_{1m}$ ($m = 1, ..., n$), and $\tilde{\boldsymbol{\phi}}$ is a diagonal matrix with elements $\exp(-\mathrm{i}\boldsymbol{k}_1'.\boldsymbol{r})$ and $\exp(-\mathrm{i}\boldsymbol{k}_m.\boldsymbol{r})$ ($m = 2, ..., n$) along the leading diagonal, $\boldsymbol{k}_1', \boldsymbol{k}_2, ..., \boldsymbol{k}_n$ being the wave vectors of the scattered particles in the respective channels.

We now suppose that the kernel is separable in the form

$$\mathbf{K}(\boldsymbol{r},\boldsymbol{r}') = \mathbf{P}^*(\boldsymbol{r})\mathbf{P}(\boldsymbol{r}'). \tag{4}$$

Then it can be readily verified that

$$\begin{aligned}\mathbf{f} = -\frac{1}{4\pi}\int\tilde{\boldsymbol{\phi}}(\boldsymbol{r})\mathbf{P}^*(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}\\ .\left\{\mathbf{I}+\int\int\mathbf{P}(\boldsymbol{r}')\mathbf{G}(\boldsymbol{r}',\boldsymbol{r}'')\mathbf{P}^*(\boldsymbol{r}'')\,\mathrm{d}\boldsymbol{r}'\,\mathrm{d}\boldsymbol{r}''\right\}^{-1}.\int\mathbf{P}(\boldsymbol{r}')\boldsymbol{\phi}(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}'.\end{aligned} \tag{5}$$

This formula is exact and, since it is entirely independent of $\mathbf{F}$ and $\tilde{\mathbf{F}}$, is given by the Schwinger variational principle using arbitrary trial functions.

For the special case of single-channel scattering (5) reduces to the result obtained by Rodberg and Thaler (1967) without employing a variational principle.

The Schwinger variational principle can also be employed to obtain an exact expression for the scattering phase shift η_l for separable non-local interactions. Thus, in the single-channel case, if we set

$$K(\mathbf{r}, \mathbf{r}') = (4\pi r r')^{-1} \sum_{l=0}^{\infty} (2l+1) K_l(r, r') P_l(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') \tag{6}$$

and separate the kernels in the form

$$K_l(r, r') = p_l^*(r) p_l(r') \tag{7}$$

we find that the Schwinger variational principle yields

$$\tan \eta_l = -\frac{k \left| \int_0^\infty r \mathrm{j}_l(kr) p_l(r)\, \mathrm{d}r \right|^2}{1 + \int_0^\infty \int_0^\infty p_l(r) G_l(r, r') p_l^*(r')\, \mathrm{d}r\, \mathrm{d}r'} \tag{8}$$

independent of the trial function employed, j_l being a spherical Bessel function and G_l being the free-particle Green function for the lth partial wave. A similar result was previously obtained by Rodberg and Thaler (1967) using a direct method. Our result is rather simpler in form than that recently derived by Cassola and Koshel (1968).

Finally, it is worth emphasizing that, in general, variational methods other than the one originated by Schwinger, do not yield exact expressions for the scattering amplitude or phase shift in the case of separable non-local interactions unless the trial function employed coincides with the exact solution.

References

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